Matheprüfung – Grenzwerte – 3C – 9/2012 – MD

1) a) $f(0) \overset{1}{=} \frac{0}{0^2+2} = 0 \qquad f(-1) \overset{1}{=} \frac{-1}{(-1)^2+2} = -\frac{1}{3} \qquad f(2a) \overset{1}{=} \frac{2a}{(2a)^2+2} \overset{1}{=} \frac{a}{1+2a^2}$

6p

$f(x+h) \overset{1}{=} \frac{x+h}{(x+h)^2+2} \overset{1}{=} \frac{x+h}{x^2+2xh+h^2+2}$

2p b) $f(-x) \overset{1}{=} \frac{-x}{(-x)^2+2} = -\frac{x}{x^2+2} \overset{1}{=} -f(-x) \Rightarrow$ ungerade!

c) D: $x^2+1 \neq 0 \Rightarrow x^2 \neq -1$ immer $\Rightarrow D \overset{1}{=} \mathbb{R}$

4p Umkehrfunktion: $y = \frac{x}{x^2+1} \Rightarrow x^2 y - x - y = 0$ (nach x!)

$\Rightarrow \quad x = \frac{1}{2y} \pm \frac{1}{2y}\sqrt{1-8y^2}$ d.h. $y \neq 0$

$1-8y^2 \geq 0 \quad 1 \geq 8y^2$

$y^2 \leq 1/8 \quad -\frac{1}{2\sqrt{2}} \leq y \leq \frac{1}{2\sqrt{2}}$ 1

Für $y=0 \Rightarrow x^2 \cdot 0 - x - 0 = 0 \Leftrightarrow -x = 0 \Leftrightarrow x = 0$ 1

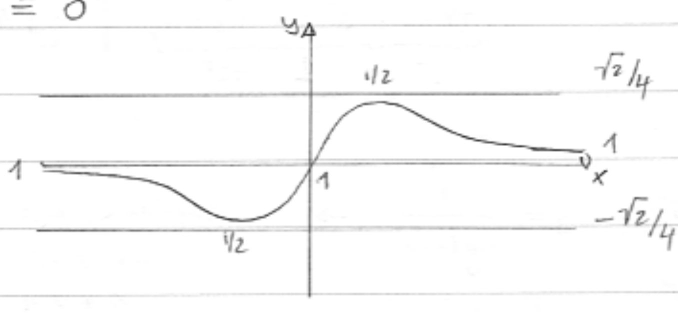
$\Rightarrow \dot{W} = \left[-\frac{1}{2\sqrt{2}} ; \frac{1}{2\sqrt{2}}\right] = \left[-\frac{\sqrt{2}}{4} ; \frac{\sqrt{2}}{4}\right]$ 1

d) $\lim\limits_{x\to+\infty} \frac{x}{\underbrace{x^2+2}_{\to x^2}} = \lim\limits_{x\to+\infty} \frac{x}{x^2} \overset{1}{=} \lim\limits_{x\to+\infty} \frac{1}{x} \overset{1}{=} 0^+$

4p

$\lim\limits_{x\to-\infty} \frac{x}{x^2+2} = \lim\limits_{x\to-\infty} \frac{1}{x} \overset{2}{=} 0^-$

4p



2) $f(g(x)) = f(cx+d) = a(cx+d)+b = acx + ad + b$ 1

6p $g(f(x)) = g(ax+b) = c(ax+b)+d = cax + cb + d$ 1

$\Rightarrow \cancel{acx} + ad + b = \cancel{acx} + bc + d \Leftrightarrow d(a-1) = b(c-1)$

1 1 1+1

3) 20p

$\lim_{x\to 2} (x^2-4x) = 2^2 - 4\cdot 2 \overset{1}{=} -4$ ‖ $\lim_{x\to -1} (x^3+2x^2-3x-4) = -1+2+3-4 \overset{1}{=} 0$

$\lim_{x\to 1} \frac{(3x-1)^2}{(x+1)^3} = \frac{2^2}{2^3} \overset{1}{=} \frac{1}{2}$ ‖ $\lim_{x\to e} \ln(x^x) = \lim_{x\to e} x\ln x = e\ln e \overset{1}{=} e$

$\lim_{x\to 1} \frac{x-1}{x^2-1} \overset{1}{=} \lim_{x\to 1} \frac{\cancel{(x-1)}}{\cancel{(x-1)}(x+1)} \overset{1}{=} \frac{1}{2}$ ‖ $\lim_{x\to +\infty} \frac{2x^2+1}{6+x-3x^2} = \lim_{x\to+\infty}^{1} \frac{\cancel{x^2}(2+\frac{1}{x^2})}{\cancel{x^2}(\frac{6}{x^2}+\frac{1}{x}-3)} \overset{1}{=} -\frac{2}{3}$ ($\to 0$, 0)

$\lim_{x\to 0} \frac{\sin^2 x}{1-\cos x} = \lim_{x\to 0}^{1} \frac{\cancel{(1-\cos x)}(1+\cos x)}{\cancel{(1-\cos x)}} = 1+1 \overset{1}{=} 2$

$\lim_{x\to 0^+}^{1/2} \frac{1}{1-e^{1/x}} = \lim_{z\to+\infty} \frac{1}{1-e^z} \overset{1}{=} 0^-$ ($-\infty$) ‖ $\lim_{x\to 0^-}^{1/2} \frac{1}{1-e^{1/x}} = \lim_{z\to-\infty} \frac{1}{1-e^z} \overset{1}{=} 1^+$ ($e^z\to 0$)

$\lim_{x\to 2} \frac{2x-3}{\sqrt{x}+\sqrt{4x+1}} = \lim_{x\to 2} \frac{2x-3}{\sqrt{x}+\sqrt{4x+1}} \cdot^{1} \frac{\sqrt{x}-\sqrt{4x+1}}{\sqrt{x}-\sqrt{4x+1}} = \lim_{x\to 2}^{1} \frac{(2x-3)(\sqrt{x}-\sqrt{4x+1})}{(x-4x-1)} = \frac{3-\sqrt{2}}{7}$ 1

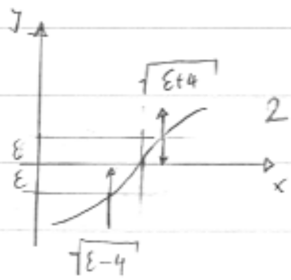
($\to +1$, $\to\sqrt{2}$, $\to 3$; $-3x+1 \to -7$)

$\lim_{x\to\alpha} \frac{\sin x-\sin\alpha}{\cos x-\cos\alpha} = \lim_{x\to\alpha}^{1} \frac{(\sin x-\sin\alpha)(\cos x+\cos\alpha)}{\underbrace{\cos^2 x-\cos^2\alpha}_{\cancel{1}-\sin^2 x-\cancel{1}+\sin^2\alpha}} = \lim_{x\to\alpha}^{1} \frac{\cancel{(\sin x-\sin\alpha)}(\cos x+\cos\alpha)}{-\cancel{(\sin x-\sin\alpha)}(\sin x+\sin\alpha)} = -\frac{2\cos\alpha}{\sin\alpha} = -\frac{1}{\tan\alpha}$ 1

4)a) 12p $\lim_{x\to+\infty} 2^{1/x} = 1$ $\quad 2^{1/x}-1 \le \varepsilon \Rightarrow 1/x \le \log_2(\varepsilon+1) \Rightarrow x > \frac{1}{\log_2(\varepsilon+1)}$ $\quad \varepsilon\to 0$, $x\to+\infty$ ✓

b) $\lim_{x\to 2^\pm} (x^2-4) = 0$

$x\to 2^+$	$x\to 2^-$
$x^2-4-0 \le \varepsilon$	$0-x^2+4 \le \varepsilon$
$x^2 \le \varepsilon+4$	$4-\varepsilon \le x^2$
$\underbrace{-\sqrt{\varepsilon+4}}_{-2^-} \le x \le \underbrace{\sqrt{\varepsilon+4}}_{+2^+}$	$\underbrace{-\sqrt{4-\varepsilon}}_{-2^+} \ge x \ge \underbrace{\sqrt{4-\varepsilon}}_{+2^-}$
2	2



5)a) 8p $y = \frac{1}{\log(1+x)} \Rightarrow \log(1+x) \ne 0$, $1+x \ne 10^0 = 1$, $x \ne 0$ 1 $\quad 1+x>0$, $x>-1$ $\quad D =]-1;\infty[\setminus \{0\}$ 1

b) $y = \frac{x^2-9}{x+3} \Rightarrow x+3 \ne 0 \Rightarrow x = -3$ $\quad D = \mathbb{R} \setminus \{-3\}$ 1

c) $y = \ln(\log_{1/10} x) \Rightarrow x>0$; $\log_{1/10} x > 0$ $\Rightarrow D =]0;1[$ 1

$x < \left(\frac{1}{10}\right)^0 = 1$ 1